On optimal Canonical Variables in the Theory of Ideal Fluid with Free Surface

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Abstract

Dynamics of ideal fluid with free surface can be effectively solved by perturbing the Hamiltonian in weak nonlinearity limit. However it is shown that perturbation theory, which includes third and fourth order terms in the Hamiltonian, results in the ill-posed equations because of short wave instability. To fix that problem we introduce the canonical Hamiltonian transform from original physical variables to new variables for which instability is absent. We found the choice of such transform is unique.

1 Introduction

The Euler equations describing dynamics of ideal fluid with free surface is a Hamiltonian system, which is especially simple if the fluid motion is potential, $v = \nabla\Phi$, where v is the fluid's velocity and Ψ is the velocity potential. In this case [1, 2] the Euler equations can be presented in the form:

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \tag{1}$$

Here $z = \eta(\mathbf{r})$ is the shape of surface, z is vertical coordinate and $\mathbf{r} = (x, y)$ are horizontal coordinates, $\Psi \equiv \Phi|_{z=\eta}$ is the velocity potential on the surface. The

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Hamiltonian H coincides with the total (potential and kinetic) energy of fluid. The Hamiltonian cannot be expressed in a closed form as a function of surface variables η, Ψ, but it can be presented by the infinite series in powers of surface steepness $|\nabla \eta|$:

$$H = H_0 + H_1 + H_2 + \ldots. \tag{2}$$

Here H_0, H_1, H_2 are quadratic, cubic and quartic terms, respectively. Equations (1), (2) are widely used used now for numerical simulation of the fluid dynamics [7, 8, 9, 10, 12, 13, 14, 15]. These simulations are performing by the use of the spectral code, at the moment a typical grid is 512×512 harmonics.

Canonical variables are used also for analytical study of the surface dynamics in the limit of small steepness. It was show [4, 5, 6] that the simplest truncation of the series (2), namely

$$H = H_0 + H_1, \tag{3}$$

leads to completely integrable model - complex Hopf equation. In framework of this approach one can develop the self-consistent theory of singularity formation in absence of gravity and capillarity for two dimensions (one vertical coordinate z and one horizontal coordinate x).

However, canonical variables η, Ψ has a weak point, which becomes clear, if we concentrate our attention on the complex Hopf equation,

$$\frac{\partial \Theta}{\partial t} = -\frac{1}{2}\left(\frac{\partial \Theta}{\partial x}\right)^2, \tag{4}$$

which comes from Eqs. (1), (3). Here

$$\Psi = Re(\Theta) \tag{5}$$

and Θ is analytic function of complex variable x in a strip $-h \leq Im(x) \leq 0$. The weak point is that Eq. (4) is ill-posed. A general complex solution of this equation is unstable with respect to grow of small short-wave perturbations. The same statement is correct with respect to more exact fourth order Hamiltonian

$$H = H_0 + H_1 + H_2, \tag{6}$$

which is used in most numerical experiments. These experiments are easy becomes unstable: to arrest instability one should include into equations strong artificial damping at high wave numbers. Even in presence of such damping one can simulate only waves of a relatively small steepness (not more that 0.15)[reference??????????].

In this Article we show that these difficulties can be fixed by a proper canonical transformation to another canonical variables. It is remarkable, but the property of nonlinear wave equation to be well- or ill-posed is *not invariant with respect to choice of the variables*.

In the present Article we demonstrate that there are *unique* canonical variables such that the Eqs. (1), (6) are well-posed. We call these variables "optimal canonical variables". We can formulate a conjecture that the optimal canonical variables exist and are unique in all orders of nonlinearity.

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2   Basic equations and Hamiltonian formalizm

Consider the dynamics incompressible ideal fluid with free surface and constant depth. Fluid occupies the region

$$-h < z < \eta(\mathbf{r}), \quad \mathbf{r} = (x, y), \tag{7}$$

where (x, y) are the horizontal coordinates and z is vertical coordinate.

Viscosity is assumed to be absent and the fluid's velocity $\mathbf{v}$ is potential one:

$$\mathbf{v} = \nabla \Phi, \tag{8}$$

where Φ is the velocity potential. Incompressibility condition,

$$\nabla \cdot \mathbf{v} = 0, \tag{9}$$

results in the Laplace Eq.

$$\triangle \Phi = 0. \tag{10}$$

The potential Φ satisfies also the Bernoulli equation:

$$\Phi_t + \frac{1}{2}\left(\nabla \Phi\right)^2 + p + gz = 0, \tag{11}$$

where p is the pressure, g is the acceleration of gravity, and we set density of fluid to unity.

There are two types of boundary conditions at free surface for Eqs. (10), (11). First is the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} = \left(\Phi_z - \nabla \eta \cdot \nabla \Phi\right)\Big|_{z=\eta} = v_n \sqrt{1 + (\nabla \eta)^2}, \tag{12}$$

where $v_n = \mathbf{n} \cdot \nabla \Phi$ is the normal component of fluid's velocity at free surface, and $\mathbf{n} = (-\nabla \eta, 1)\left[1 + (\nabla \eta)^2\right]^{-1/2}$ is the interface normal vector.

Second is the dynamic boundary condition at free surface

$$p\big|_{z=\eta} = \sigma \nabla \cdot \frac{\nabla \eta}{\sqrt{1 + (\nabla \eta)^2}}, \tag{13}$$

where σ is the surface tension coefficient which determines the jump of pressure at free surface from zero value out of the fluid to $p\big|_{z=\eta}$ value according to Eq. (13).

Boundary condition at the bottom is

$$\Phi_z\big|_{z=-h}. \tag{14}$$

Eqs. (10) − (14) form a closed set of equations to determine the dynamics of free surface.

The total energy, H, of the fluid consists of kinetic energy, T, and potential energy, U:

$$H = T + U, \tag{15}$$

$$T = \frac{1}{2} \int d\mathbf{r} \int_{-h}^{\eta} (\nabla \Phi)^2 dz, \tag{16}$$

$$U = \frac{1}{2}g \int \eta^2 \, d\mathbf{r} + \sigma \int \left[\sqrt{1 + (\nabla \eta)^2} - 1 \right] d\mathbf{r}. \tag{17}$$

It is a convenient to introduce the value of velocity potential at interface with the boundary conditions

$$\Phi \big|_{z=\eta} \equiv \Psi(\mathbf{r}, t). \tag{18}$$

It was shown in Ref. [2] that free surface problem $(10) - (14)$ can be written in the Hamiltonian form (1), with the Hamiltonian H defined in (15).

Fourier transform,

$$\Psi_{\mathbf{k}} = \frac{1}{2\pi} \int \exp(-i\mathbf{k} \cdot \mathbf{r}) \Psi(\mathbf{r}) d\mathbf{r}, \tag{19}$$

is the canonical transform which conserves the Hamiltonian structure and Eqs. (1) take the following form:

$$\frac{\partial \eta_{\mathbf{k}}}{\partial t} = \frac{\delta H}{\delta \Psi_{-\mathbf{k}}}, \quad \frac{\partial \Psi_{\mathbf{k}}}{\partial t} = -\frac{\delta H}{\delta \eta_{-\mathbf{k}}}, \quad \eta_{\mathbf{k}}^* = \eta_{-\mathbf{k}}, \quad \Psi_{\mathbf{k}}^* = \Psi_{-\mathbf{k}}. \tag{20}$$

3　Weak nonlinearity

If typical slope of free surface is small, $|\nabla \eta| \ll 1$, the Hamiltonian H is can be series expanded (see Eq. (2)) in powers of stepness $|\nabla \eta|$ which gives [2, 3]:

$$H_0 = \frac{1}{2} \int \left\{ A_k |\Psi_{\mathbf{k}}|^2 + B_k |\eta_{\mathbf{k}}|^2 \right\} d\mathbf{k},$$

$$A_k = k \tanh(kh), \ B_k = g + \sigma k^2, \ k = |\mathbf{k}|, \tag{21}$$

$$H_1 = \frac{1}{4\pi} \int L_{\mathbf{k}_1, \mathbf{k}_2}^{(1)} \Psi_{\mathbf{k}_1} \Psi_{\mathbf{k}_2} \eta_{\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \tag{22}$$

$$H_2 = \frac{1}{2(2\pi)^2} \int \left[L_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4}^{(2)} \Psi_{\mathbf{k}_1} \Psi_{\mathbf{k}_2} - \frac{\sigma}{4} (\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_3 \cdot \mathbf{k}_4) \eta_{\mathbf{k}_1} \eta_{\mathbf{k}_2} \right]$$
$$\times \eta_{\mathbf{k}_3} \eta_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4, \tag{23}$$

where matrix elements are given by

$$L_{\mathbf{k}_1, \mathbf{k}_2}^{(1)} = -\mathbf{k}_1 \cdot \mathbf{k}_2 - A_1 A_2,$$

$$L_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4}^{(2)} = \frac{1}{4} A_1 A_2 \left(A_{1+3} + A_{2+3} + A_{1+4} + A_{2+4} \right)$$
$$- \frac{1}{2} (k_1^2 A_2 + k_2^2 A_1), \quad A_j \equiv A_{k_j}. \tag{24}$$

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The corresponding dynamical equations follow from $(1),(6),(21),(22),(23)$:

$$\frac{\partial \Psi}{\partial t} = -g\eta + \sigma\Delta\eta + \frac{1}{2}\left[(\hat{A}\Psi)^2 - (\nabla\Psi)^2\right] - (\hat{A}\Psi)\hat{A}[\eta(\hat{A}\Psi)]$$
$$-(\Delta\Psi)(\hat{A}\Psi)\eta - \frac{\sigma}{2}\nabla\cdot\left[\nabla\eta(\nabla\eta\cdot\nabla\eta)\right], \qquad (25)$$

$$\frac{\partial \eta}{\partial t} = \hat{A}\Psi - \nabla\cdot\left[(\nabla\Psi)\eta\right] - \hat{A}[\eta\hat{A}\Psi] + \hat{A}\left\{\eta\hat{A}[\eta\hat{A}\Psi]\right\}$$
$$+\frac{1}{2}\Delta[\eta^2\hat{A}\Psi] + \frac{1}{2}\hat{A}[\eta^2\Delta\Psi], \qquad (26)$$

where $\hat{A}$ is the linear integral operator which corresponds to multiplication on $k\tanh(kh)$ in Fourier space. For two dimensional flow, $\Psi(x,y) = \Psi(x)$, $\eta(x,y) = \eta(x)$, this operator is given by

$$\hat{A} = -\frac{\partial}{\partial x}\hat{R} \qquad (27)$$

$$\hat{R}f(x) = \frac{1}{2h}P.V.\int_{-\infty}^{+\infty}\frac{f(x')}{\sinh\left[(x'-x)\pi/(2h)\right]}\,dx' \qquad (28)$$

where $P.V.$ means Cauchy principal value of integral. In the limiting case of infinitely deep water, $h\to\infty$, operator $\hat{A}$ turns into operator $\hat{k}$

$$\lim_{h\to\infty}\hat{A} = \hat{k} \qquad (29)$$

which corresponds to multiplication on $|\mathbf{k}|$ in Fourier space while operator $\hat{R}$ for two-dimentsional flow turns into the Hilbert transform:

$$\lim_{h\to\infty}\hat{R} = \hat{H}, \quad Hf(x) = \frac{1}{\pi}P.V.\int_{-\infty}^{+\infty}\frac{f(x')}{x'-x}dx'. \qquad (30)$$

$\hat{H}$ can be also interpreted as a Fourier transform of $-i\,\text{sign}(k)$.

If one neglects gravity and surface tension, $g = 0$, $\sigma = 0$, than Eqs. $(1),(2)$ at leading order over small parameter $|\nabla\eta|$ result in[5, 4, 6]

$$\frac{\partial \eta}{\partial t} = \hat{A}\Psi, \qquad (31)$$

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2}\left[(\hat{A}\Psi)^2 - (\nabla\Psi)^2\right]. \qquad (32)$$

Remarkable feature of Eqs. $(31),(32)$ is that the second Eq. (32) does not depend on η thus one can first solve (32) and then find η from Eq. (31). Substitution $\Theta \equiv \Psi + i\hat{R}\Psi$ into Eq. (32) results in the complex Hopf Eq. (4) for two-dimensional flow [6] which is completely integrable.

Both Eqs. (32) and (4) are ill-posed because they have short wavelength instability which determines as follows. E.g. we can analyze Eq. (32). Take Ψ in the form

$$\Psi = \Psi_0 + \left(\Psi_1 e^{i\mathbf{k_0}\cdot\mathbf{r}+\nu t} + c.c.\right), \qquad (33)$$

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where $\Psi_0(\mathbf{r}, t)$ is a solution of Eq. (32), Ψ_1 is the amplitude of small perturbation, and *c.c.* means complex conjugation. Then in the limit $|\mathbf{k}_0| \to \infty$ Ψ_0 evolve very slow in space compare to $e^{i\mathbf{k}_0 \cdot \mathbf{r} + \nu t}$ and we get the dispersion relation for small perturbations:

$$\nu = A_{k_0}\hat{A}\Psi_0 - i\mathbf{k}_0 \cdot \nabla\Psi_0 \tag{34}$$

which describes instability for $Re(\nu) = A_{k_0}\hat{A}\Psi_0 > 0$. For general initial condition such instability region always exists. Instability growth rate, $Re(\nu)$ grows as $|\mathbf{k}_0|$ increases.

4 Ill-posedness of the fourth-order Hamiltonian

Consider now a more general case of nonzero g and σ and take into account all terms in the Hamiltonian up to forth order, i.e. consider full Eqs. (25). Similar to previous section we linearize Eqs. (25) using ansatz

$$\eta = \eta_0 + \left(\eta_1 e^{i\mathbf{k}_0 \cdot \mathbf{r} + \nu t} + c.c.\right),$$

$$\Psi = \Psi_0 + \left(\Psi_1 e^{i\mathbf{k}_0 \cdot \mathbf{r} + \nu t} + c.c.\right), \tag{35}$$

were $\eta_0(\mathbf{r}, t)$, $\Psi_0(\mathbf{r}, t)$ are solutions of (25), and get for $|\mathbf{k}_0| \to \infty$:

$$\nu = \frac{\mu}{2} + \frac{1}{2}\Big(\mu^2 + 2\big[A_{k_0} + (A_{k_0}^2 - k^2)(-1 + A_{k_0}\eta_0)\eta_0\big]$$

$$\times \big[-2g + \{3(\nabla\eta_0)^2 - 2\}k^2\sigma\big]\Big)^{1/2},$$

$$\mu = -i\mathbf{k}_0 \cdot \nabla\Psi_0 + \big[A_{k_0} + (k_0^2 - A_{k_0}^2)\eta_0\big]\hat{A}\Psi_0. \tag{36}$$

Instability growth rate take more compact form for infinite depth fluid as $\lim_{h \to \infty} A_k = k$:

$$\nu = \frac{\mu}{2} + \frac{1}{2}\Big(\mu^2 + 2k_0\big[-2g + \{3(\nabla\eta_0)^2 - 2\}k^2\sigma\big]\Big)^{1/2},$$

$$\mu = -i\mathbf{k}_0 \cdot \nabla\Psi_0 + k_0\hat{k}\Psi_0. \tag{37}$$

For general initial condition with arbitrary depth h the instability region $Re(\nu) = A_{k_0}\hat{A}\Psi_0 > 0$ always exists. We conclude that full fourth order Hamiltonian does not prevent short wavelength instability and Eqs. (25) are ill-posed.

Ill-posedness of Eqs. $(20), (21) - (23)$ (or, equivalently, Eqs. (25)) makes them difficult for numerical analysis. There a few ways to cope with that problem. One way is to introduce artificial damping for short wavelengths, i.e. to replace Eqs. (20) by

$$\frac{\partial \eta_{\mathbf{k}}}{\partial t} = \frac{\delta H}{\delta \Psi_{-\mathbf{k}}} + \gamma_1(k)\eta_{\mathbf{k}}, \quad \frac{\partial \Psi_{\mathbf{k}}}{\partial t} = -\frac{\delta H}{\delta \eta_{-\mathbf{k}}} + \gamma_2(k)\Psi_{\mathbf{k}}, \tag{38}$$

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where functions $\gamma_1(k)$, $\gamma_2(k)$ are zero for small and intermediate values of k but they tend to $-\infty$ for $k \to \infty$. Second way is to introduce finite viscosity of the fluid. However in that case the Hamiltonian does not conserve and we can not use the Hamiltonian formalism. In this Paper we use third way which is to introduce new canonical transform of variables η, Ψ to remove short wavelength instability. Advantage of this methods is that, in contrast to first way, we do not introduce any artificial damping, and, in contrast to second way, we preserve the Hamiltonian formulation of free surface dynamics problem.

5 Canonical transform

Canonical transform from variables Ψ, η to new variables R, ξ is determined by the generating function S :

$$S = \int R_{\mathbf{k}} \eta_{-\mathbf{k}} dk + \frac{1}{8\pi} \int A_3 \eta_{\mathbf{k}_1} \eta_{\mathbf{k}_2} R_{\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) dk_1 dk_2 dk_3$$

$$+\frac{1}{4(2\pi)^2} \int V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} R_{\mathbf{k}_1} \eta_{\mathbf{k}_2} \eta_{\mathbf{k}_3} \eta_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) dk_1 dk_2 dk_3 dk_4, \quad (39)$$

$$\Psi_{\mathbf{k}} = \frac{\delta S}{\delta \eta_{-\mathbf{k}}} = R_{\mathbf{k}} + \frac{1}{4\pi} \int A_1 R_{\mathbf{k}_1} \eta_{\mathbf{k}_2} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) dk_1 dk_2$$

$$+\frac{3}{4(2\pi)^2} \int V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,-\mathbf{k}} R_{\mathbf{k}_1} \eta_{\mathbf{k}_2} \eta_{\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}) dk_1 dk_2 dk_3, \quad (40)$$

$$\xi_{\mathbf{k}} = \frac{\delta S}{\delta R_{-\mathbf{k}}} = \eta_{\mathbf{k}} + \frac{1}{8\pi} \int A_k \eta_{\mathbf{k}_1} \eta_{\mathbf{k}_2} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) dk_1 dk_2$$

$$+\frac{1}{4(2\pi)^2} \int V_{-\mathbf{k},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \eta_{\mathbf{k}_2} \eta_{\mathbf{k}_3} \eta_{\mathbf{k}_4} \delta(\mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}) dk_2 dk_3 dk_4, \quad (41)$$

where $V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}$ is the symmetric function of $\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$. This is the most general form of canonical transform up to terms of fourth order. The only condition which we use here is that S is chosen to be linear functional R to preserve the quadratic dependence of the Hamiltonian on canonical momentum R.

η can be found from Eq. (41) as the functional of ξ by iterations (here and below we take into account only corrections up to the fourth order in the Hamiltonian):

$$\eta_{\mathbf{k}} = \xi_{\mathbf{k}} - \frac{1}{8\pi} \int A_k \xi_{\mathbf{k}_1} \xi_{\mathbf{k}_2} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) dk_1 dk_2 + \frac{1}{8(2\pi)^2}$$

$$\times \int \left[A_k A_{1+2} - 2V_{-\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \right] \xi_{\mathbf{k}_1} \xi_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}) dk_1 dk_2 dk_3, \quad (42)$$

Eqs. (40), (42) give:

$$\Psi_{\mathbf{k}} = R_{\mathbf{k}} + \frac{1}{4\pi} \int A_1 R_{\mathbf{k}_1} \xi_{\mathbf{k}_2} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) dk_1 dk_2 +$$

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$$\frac{1}{8(2\pi)^2} \int \left[-A_1 A_{2+3} + 6V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,-\mathbf{k}} \right] R_{\mathbf{k}_1} \xi_{\mathbf{k}_2} \xi_{\mathbf{k}_3}$$

$$\times \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k})dk_1 dk_2 dk_3, \tag{43}$$

Using Eqs. $(21),(22),(23),(40),(42)$ we get:

$$H_0 = \frac{1}{2} \int \left\{ A_k |R_{\mathbf{k}}|^2 + B_k |\xi_{\mathbf{k}}|^2 \right\} dk, \tag{44}$$

$$H_1 = \frac{1}{4\pi} \int \left[-(\mathbf{k}_1 \cdot \mathbf{k}_2) R_{\mathbf{k}_1} R_{\mathbf{k}_2} - \frac{1}{6}(A_1 B_1 + A_2 B_2 + A_3 B_3)\xi_{\mathbf{k}_1} \xi_{\mathbf{k}_2} \right]$$

$$\times \xi_{\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)dk_1 dk_2 dk_3, \tag{45}$$

$$H_2 = \frac{1}{8(2\pi)^2} \int \left\{ (\mathbf{k}_1 \cdot \mathbf{k}_2)(A_{1+2} - A_1 - A_2) - k_1^2 A_2 - k_2^2 A_1 \right.$$

$$+ \frac{1}{4} A_1 A_2 [A_{1+3} + A_{2+3} + A_{1+4} + A_{2+4}] + 3[A_1 V_{\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_1}$$

$$\left. + A_2 V_{\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_2}] \right\} R_{\mathbf{k}_1} R_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \xi_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)dk_1 dk_2 dk_3 dk_4$$

$$+ \frac{1}{8(2\pi)^2} \int \left\{ -\sigma(\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_3 \cdot \mathbf{k}_4) + \frac{1}{4} A_{1+2}^2 B_{1+2} + A_3 B_3 A_{1+2} \right.$$

$$\left. - 2B_1 V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \right\} \xi_{\mathbf{k}_1} \xi_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \xi_{\mathbf{k}_4}$$

$$\times \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)dk_1 dk_2 dk_3 dk_4, \tag{46}$$

Canonical transform conserves the Hamiltonian structure so the dynamical equations in new variables R, ξ are given by:

$$\frac{\partial \xi}{\partial t} = \frac{\delta H}{\delta R}, \quad \frac{\partial R}{\partial t} = -\frac{\delta H}{\delta \xi}. \tag{47}$$

6 From complex to real Hopf equation

We chose the cubic term of the generation function S in such a way to remove linear instability at leading order. Similar to Eqs. $(31),(32)$, we get from Eqs. $(44),(45),(47)$ at leading order of small parameter $|\nabla \xi|$:

$$\frac{\partial \xi}{\partial t} = \hat{A}R, \tag{48}$$

$$\frac{\partial R}{\partial t} = -\frac{1}{2}(\nabla R)^2. \tag{49}$$

Thus instead of the complex Hopf Eq. (4) (or Eq. (32)) we got real Burgers Eq. (49) for new canonical variable R. It is important that the real Burgers Eq. is well-posed.

Additional advantage of (49) is that it can be integrated by the the method of characteristic not only in two dimensions as Eq. (4) but for three dimensional flow also.

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7 Removal of instability from fourth order term

Next step is to remove instability from the fourth order terms in the Hamiltonian (46) by proper choice of matrix element V. We can take $V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}$ in the following form:

$$V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} = \alpha_1 k_1^2 + \alpha_2 A_1 (A_{2+3} + A_{2+4} + A_{3+4}), \tag{50}$$

where α_1, α_2 are the real constants.

The dynamical equations, as follows from (44), (45), (46), (47), (50) are

$$\frac{\partial R}{\partial t} = -\hat{B}\xi - \frac{1}{2}(\nabla R)^2 + \frac{1}{2}\xi\hat{A}\hat{B}\xi + \frac{1}{4}\hat{A}\hat{B}\xi^2 + \frac{1}{4}\xi\hat{A}(\nabla R)^2$$

$$-\frac{1}{2}\xi\nabla R \cdot \nabla\hat{A}R - \frac{1}{2}(1-3\alpha_1)\xi(\triangle R)\hat{A}R - \left(\frac{1}{4}+3\alpha_2\right)(\hat{A}R)\hat{A}(\xi\hat{A}R)$$

$$-\frac{3\alpha_2}{2}\xi\hat{A}\big[(\hat{A}R)^2\big] - \frac{\sigma}{2}\nabla\cdot\Big[\nabla\xi(\nabla\xi\cdot\nabla\xi)\Big] - \frac{1}{8}\xi\hat{A}^2(\hat{B}\xi^2)$$

$$-\frac{1}{8}(\hat{A}\hat{B}\xi)\hat{A}\xi^2 - \frac{1}{8}\hat{A}\hat{B}(\xi\hat{A}\xi^2) - \frac{1-6\alpha_2}{4}\xi\hat{A}(\xi\hat{A}\hat{B}\xi) - \frac{3}{4}\alpha_1\xi^2\triangle\hat{B}\xi$$

$$+\frac{3}{4}\alpha_2(\hat{A}\hat{B}\xi)\hat{A}\xi^2 - \frac{\alpha_1}{4}\triangle\hat{B}\xi^3 + \frac{3\alpha_2}{4}\hat{A}\hat{B}(\xi\hat{A}\xi^2), \tag{51}$$

$$\frac{\partial\xi}{\partial t} = \hat{A}R - \nabla\cdot[(\nabla R)\xi] + \frac{1}{4}\nabla\cdot\Big[(\nabla R)\hat{A}\xi^2\Big] - \frac{1}{4}\nabla\cdot\hat{A}(\xi^2\nabla R)$$

$$-\frac{1}{4}\nabla\cdot(\xi^2\hat{A}\nabla R) + \frac{1-3\alpha_1}{4}\triangle(\xi^2\hat{A}R) + \frac{1-3\alpha_1}{4}\hat{A}(\xi^2\triangle R)$$

$$+\left(\frac{1}{4}+3\alpha_2\right)\hat{A}\Big[\xi\hat{A}(\xi\hat{A}R)\Big] + \frac{3}{2}\alpha_2\hat{A}\Big[(\hat{A}\xi^2)(\hat{A}R)\Big], \tag{52}$$

where $\hat{B} \equiv g - \sigma\triangle$.

Using ansatz

$$\xi = \xi_0 + \left(\xi_1 e^{i\mathbf{k}_0\cdot\mathbf{r}+\nu t} + c.c.\right),$$

$$R = R_0 + \left(R_1 e^{i\mathbf{k}_0\cdot\mathbf{r}+\nu t} + c.c.\right), \tag{53}$$

one can linearize Eqs. (51), (51) on a background of solution $\xi_0(\mathbf{r},t)$, $R_0(\mathbf{r},t)$ a, and get for $|\mathbf{k}_0| \to \infty$:

$$\nu = \frac{\mu_1}{2} + \frac{1}{2}\left(\mu_2^2 + \mu_3\right)^{1/2}, \tag{54}$$

$$\mu_1 = -i\mathbf{k}_0\cdot\nabla R_0 + \frac{i}{2}A_{k_0}\xi_0\mathbf{k}_0\cdot\nabla R_0$$

$$-\frac{1}{4}[2(3\alpha_1-1)k_0^2 + (1+12\alpha_2)A_{k_0}^2]\xi_0\hat{A}R_0,$$

$$\mu_2 = i\mathbf{k}_0\cdot\nabla R_0 + \frac{i}{2}A_{k_0}\xi_0\mathbf{k}_0\cdot\nabla R_0$$

$$+\frac{1}{4}[2(3\alpha_1-1)k_0^2 + (1+36\alpha_2)A_{k_0}^2]\xi_0\hat{A}R_0,$$

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$$\mu_3 = \frac{1}{8}\Big\{ 2\big(B_{k_0}\big[4A_{k_0}\xi_0 - 4 + 3A_{k_0}^2(4\alpha_2 - 1)\xi_0^2 + 6\alpha_1 k_0^2 \xi_0^2\big]$$

$$+6k_0^2\sigma(\nabla\xi_0)^2\big) + A_{k_0}(6\alpha_2 - 1)B_{k_0}\hat{A}\xi_0^2\Big\}\Big\{4k_0^2\xi_0$$

$$+(1 + 12\alpha_2)A_{k_0}^3\xi_0^2 + (4 + 6\alpha_1 k_0^2\xi_0^2)A_{k_0} + (6A_{k_0}^2\alpha_2 - k_0^2)\hat{A}\xi_0^2\Big\}. \tag{55}$$

To avoid instability it is necessary to have purely imaginary ν. Necessary condition for that is that the expression under square root, $\mu_2^2 + \mu_3$, in Eq. (54) should have zero imaginary part. It follows form Eqs. (54) that μ_3 is always real, and μ_2^2 is real provided

$$2(3\alpha_1 - 1)k_0^2 + (1 + 36\alpha_2)A_{k_0}^2 = 0. \tag{56}$$

Second condition to have purely imaginary ν is $Re(\mu_1) = 0$, which gives the second condition

$$2(3\alpha_1 - 1)k_0^2 + (1 + 12\alpha_2)A_{k_0}^2 = 0. \tag{57}$$

It follows from Eqs. (56), (57) that in the limit $k_0 \to \infty$ (remember that $A_{k_0} \to k_0$ in that limit) the system (44), (45), (46), (47), (50) is well-posed provided

$$\alpha_1 = 1/6, \quad \alpha_2 = 0, \tag{58}$$

which gives from (46), (50) the well-posed fourth-order Hamiltonian:

$$H_2 = \frac{1}{8(2\pi)^2} \int \Big\{ (\mathbf{k}_1 \cdot \mathbf{k}_2)(A_{1+2} - A_1 - A_2) - \frac{1}{2}(k_1^2 A_2 + k_2^2 A_1)$$

$$+\frac{1}{4}A_1 A_2\big[A_{1+3} + A_{2+3} + A_{1+4} + A_{2+4}\big]\Big\}$$

$$\times R_{\mathbf{k}_1} R_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \xi_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4$$

$$+\frac{1}{8(2\pi)^2} \int \Big\{ -\sigma(\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_3 \cdot \mathbf{k}_4) + \frac{1}{4}A_{1+2}^2 B_{1+2} + k_3 B_3 A_{1+2}$$

$$-\frac{1}{3}B_1 k_1^2 \Big\} \xi_{\mathbf{k}_1} \xi_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \xi_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4, \tag{59}$$

This Hamiltonian is well-posed for any g, σ (including case $g = \sigma = 0$).

To find dynamics of free surface, one can solve Eqs. for R, ξ using Eqs. (47), (44), (45), (59). This is the main result of this Article. To recover physical variables Ψ, η from given R, ξ one can use Eqs. (42), (43), (50), (58).

As follows from Eq. (58), new canonical variables ξ, R are uniquely determined from the condition of well-posedness of dynamical Eqs. in new variables up to the fourth order in the Hamiltonian. We refer to these variables as *optimal canonical variables*. For some extent similar results were obtained by Dyachenko and Shamin [18] for particular case of two-dimensional flow. We conjecture that the optimal canonical variables exist and are unique in all orders of nonlinearity.

8 Special cases

There are a number of important special cases of optimal canonical variables.

8.1 Deep water limit

For $h \to \infty$ Eqs. (44), (45), (59) take the form

$$H_0 = \frac{1}{2} \int \left\{ k|R_{\mathbf{k}}|^2 + B_k |\xi_{\mathbf{k}}|^2 \right\} d\mathbf{k}, \quad (60)$$

$$H_1 = \frac{1}{4\pi} \int \left[-(\mathbf{k}_1 \cdot \mathbf{k}_2) R_{\mathbf{k}_1} R_{\mathbf{k}_2} - \frac{1}{6}(k_1 B_1 + k_2 B_2 + k_3 B_3)\xi_{\mathbf{k}_1}\xi_{\mathbf{k}_2} \right]$$
$$\times \xi_{\mathbf{k}_3}\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \quad (61)$$

$$H_2 = \frac{1}{8(2\pi)^2} \int \left\{ (\mathbf{k}_1 \cdot \mathbf{k}_2)(|\mathbf{k}_1 + \mathbf{k}_2| - k_1 - k_2) - \frac{1}{2}(k_1^2 k_2 + k_2^2 k_1) \right.$$
$$\left. + \frac{1}{4}k_1 k_2 \left[|\mathbf{k}_1 + \mathbf{k}_3| + |\mathbf{k}_2 + \mathbf{k}_3| + |\mathbf{k}_1 + \mathbf{k}_4| + |\mathbf{k}_2 + \mathbf{k}_4| \right] \right\}$$
$$\times R_{\mathbf{k}_1} R_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \xi_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4$$
$$+ \frac{1}{8(2\pi)^2} \int \left\{ -\sigma(\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_3 \cdot \mathbf{k}_4) + \frac{1}{4}|\mathbf{k}_1 + \mathbf{k}_2|^2 B_{1+2} + k_3 B_3 |\mathbf{k}_1 + \mathbf{k}_2| \right.$$
$$\left. - \frac{1}{3}B_1 k_1^2 \right\} \xi_{\mathbf{k}_1} \xi_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \xi_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4. \quad (62)$$

8.1.1 Zero gravity and capillarity $g = \sigma = 0$

$$H_0 = \frac{1}{2} \int k|R_{\mathbf{k}}|^2 d\mathbf{k}, \quad (63)$$

$$H_1 = -\frac{1}{4\pi} \int (\mathbf{k}_1 \cdot \mathbf{k}_2) R_{\mathbf{k}_1} R_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \quad (64)$$

$$H_2 = \frac{1}{8(2\pi)^2} \int \left\{ (\mathbf{k}_1 \cdot \mathbf{k}_2)(|\mathbf{k}_1 + \mathbf{k}_2| - k_1 - k_2) - \frac{1}{2}(k_1^2 k_2 + k_2^2 k_1) \right.$$
$$\left. + \frac{1}{4}k_1 k_2 \left[|\mathbf{k}_1 + \mathbf{k}_3| + |\mathbf{k}_2 + \mathbf{k}_3| + |\mathbf{k}_1 + \mathbf{k}_4| + |\mathbf{k}_2 + \mathbf{k}_4| \right] \right\}$$
$$\times R_{\mathbf{k}_1} R_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \xi_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4. \quad (65)$$

8.1.2 Zero gravity, $g = 0$, and nonzero capillarity $\sigma \neq 0$

$$H_0 = \frac{1}{2} \int \left\{ k|R_{\mathbf{k}}|^2 + \sigma k^2 |\xi_{\mathbf{k}}|^2 \right\} d\mathbf{k}, \quad (66)$$

$$H_1 = \frac{1}{4\pi} \int \left[-(\mathbf{k}_1 \cdot \mathbf{k}_2) R_{\mathbf{k}_1} R_{\mathbf{k}_2} - \frac{\sigma}{6}(k_1^3 + k_2^3 + k_3^3)\xi_{\mathbf{k}_1}\xi_{\mathbf{k}_2} \right]$$
$$\times \xi_{\mathbf{k}_3}\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \quad (67)$$

$$H_2 = \frac{1}{8(2\pi)^2} \int \left\{ (\mathbf{k}_1 \cdot \mathbf{k}_2)(|\mathbf{k}_1 + \mathbf{k}_2| - k_1 - k_2) - \frac{1}{2}(k_1^2 k_2 + k_2^2 k_1) \right.$$
$$\left. + \frac{1}{4}k_1 k_2 \left[|\mathbf{k}_1 + \mathbf{k}_3| + |\mathbf{k}_2 + \mathbf{k}_3| + |\mathbf{k}_1 + \mathbf{k}_4| + |\mathbf{k}_2 + \mathbf{k}_4| \right] \right\}$$
$$\times R_{\mathbf{k}_1} R_{\mathbf{k}_2} \xi_{\mathbf{k}_3} \xi_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4$$

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$$+\frac{\sigma}{8(2\pi)^2}\int\Big\{-(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_3\cdot\mathbf{k}_4)+\frac{1}{4}|\mathbf{k}_1+\mathbf{k}_2|^4+k_3^3|\mathbf{k}_1+\mathbf{k}_2|$$

$$-\frac{1}{3}k_1^4\Big\}\xi_{\mathbf{k}_1}\xi_{\mathbf{k}_2}\xi_{\mathbf{k}_3}\xi_{\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4. \qquad (68)$$

8.1.3 Nonzero gravity, $g\neq0$, and zero capillarity $\sigma=0$

$$H_0=\frac{1}{2}\int\Big\{k|R_{\mathbf{k}}|^2+g|\xi_{\mathbf{k}}|^2\Big\}d\mathbf{k},\quad B_k=g+\sigma k^2, \qquad (69)$$

$$H_1=\frac{1}{4\pi}\int\Big[-(\mathbf{k}_1\cdot\mathbf{k}_2)R_{\mathbf{k}_1}R_{\mathbf{k}_2}-\frac{1}{6}(k_1g+k_2g+k_3g)\xi_{\mathbf{k}_1}\xi_{\mathbf{k}_2}\Big]$$

$$\times\xi_{\mathbf{k}_3}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3, \qquad (70)$$

$$H_2=\frac{1}{8(2\pi)^2}\int\Big\{(\mathbf{k}_1\cdot\mathbf{k}_2)(|\mathbf{k}_1+\mathbf{k}_2|-k_1-k_2)-\frac{1}{2}(k_1^2k_2+k_2^2k_1)$$

$$+\frac{1}{4}k_1k_2\big[|\mathbf{k}_1+\mathbf{k}_3|+|\mathbf{k}_2+\mathbf{k}_3|+|\mathbf{k}_1+\mathbf{k}_4|+|\mathbf{k}_2+\mathbf{k}_4|\big]\Big\}$$

$$\times R_{\mathbf{k}_1}R_{\mathbf{k}_2}\xi_{\mathbf{k}_3}\xi_{\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4$$

$$+\frac{1}{8(2\pi)^2}\int\Big\{-\sigma(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_3\cdot\mathbf{k}_4)+\frac{1}{4}|\mathbf{k}_1+\mathbf{k}_2|^2g+k_3g|\mathbf{k}_1+\mathbf{k}_2|$$

$$-\frac{1}{3}gk_1^2\Big\}\xi_{\mathbf{k}_1}\xi_{\mathbf{k}_2}\xi_{\mathbf{k}_3}\xi_{\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4. \qquad (71)$$

8.2 Shallow water limit

Shallow water limit corresponds to $kh\to0$. In that limit $A_k\to k^2h$. Eqs. $(44),(45),(59)$ take the form

$$H_0=\frac{1}{2}\int\Big\{k^2h|R_{\mathbf{k}}|^2+B_k|\xi_{\mathbf{k}}|^2\Big\}d\mathbf{k}, \qquad (72)$$

$$H_1=\frac{1}{4\pi}\int\Big[-(\mathbf{k}_1\cdot\mathbf{k}_2)R_{\mathbf{k}_1}R_{\mathbf{k}_2}-\frac{h}{6}(k_1^2B_1+k_2^2B_2+k_3^2B_3)\xi_{\mathbf{k}_1}\xi_{\mathbf{k}_2}\Big]$$

$$\times\xi_{\mathbf{k}_3}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3, \qquad (73)$$

$$H_2=\frac{h}{8(2\pi)^2}\int\Big\{2(\mathbf{k}_1\cdot\mathbf{k}_2)^2-\frac{1}{2}(k_1^2A_2+k_2^2A_1)$$

$$+\frac{h^2}{4}k_1^2k_2^2\big[|\mathbf{k}_1+\mathbf{k}_3|^2+|\mathbf{k}_2+\mathbf{k}_3|^2+|\mathbf{k}_1+\mathbf{k}_4|^2+|\mathbf{k}_2+\mathbf{k}_4|^2\big]\Big\}$$

$$\times R_{\mathbf{k}_1}R_{\mathbf{k}_2}\xi_{\mathbf{k}_3}\xi_{\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4$$

$$+\frac{1}{8(2\pi)^2}\int\Big\{-\sigma(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_3\cdot\mathbf{k}_4)+\frac{h^2}{4}|\mathbf{k}_1+\mathbf{k}_2|^4B_{1+2}$$

$$+k_3B_3|\mathbf{k}_1+\mathbf{k}_2|^2-\frac{h}{3}B_1k_1^2\Big\}$$

$$\times\xi_{\mathbf{k}_1}\xi_{\mathbf{k}_2}\xi_{\mathbf{k}_3}\xi_{\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4. \qquad (74)$$

However conditions $(56),(57)$ can not be simultaneously satisfied for shallow water. Thus shallow water problem remains ill-posed even for optimal variables

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because of short wavelength instability coming from the fourth order term in the Hamiltonian although instability from third-order term is removed by canonical transform (40), (41). Actually there is no big surprise in that because we need to choose carefully the order of taking the limits $kh \to 0$ and $k_0 h \to \infty$, where k is the typical wavevector of surface motion and $\mathbf{k}_0$ is a wavevector of short scale perturbation. It means that we need first to solve dynamics of water with finite depth h which is well-posed problem in optimal canonical variables, and only after that we should take limit $kh \to 0$.

9   Conclusion

In conclusion, we found optimal canonical variables for which the water wave problem is well-posed in the approximation which keeps terms up to fourth order in the Hamiltonian. The important question remain open if it is possible to make water wave equations well-posed by proper choice of canonical transform for higher-order corrections (fifth and higher order). We conjecture that such optimal canonical variables exist and are unique in all orders of nonlinearity.

10   Acknowledgements

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